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THE FOCAL CIRCLES OF SPHERICAL CONICS.

BY THE RIGHT REV. CHARLES GRAVES, D.D., Lord Bishop of Limerick.

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IX.

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[Read June 24, 1889.]

In June, 1888, a contribution appeared in *Hermathena*, in which I established, geometrically, many of the properties of the Focal Circles of Plane Conics. A single principle was involved in their deduction, viz. the motion of a sphere inside a tangent right cone. In addition, some of the properties of the Focal Circles of Spherical Conics were derived from the same idea.

It is now proposed to enter more fully into the theory of this latter class of circles, and to show how their properties may be developed by the aid of Spherical Co-ordinates. I begin by discussing the following problem:—

1. To determine the conditions which limit the position and radius of a circle, such that the trigonometrical tangent of the arc drawn to touch it, from any point on a sphero-conic, shall always be a rational linear function of the trigonometrical tangent of the abscissa.

Let x, y be the spherical co-ordinates* of any point on the conic, and x', y' those of the spherical centre of the small circle, t the tangent arc from x, y to the circle, and δ the angular distance between x, y and x', y'. Then, if r is the radius of the circle,

$$\cos t = \frac{\cos \delta}{\cos r} = \frac{1 + xx' + yy'}{\cos r \sqrt{(1 + x'^2 + y'^2)(1 + x^2 + y^2)}},$$

or,

$$\tan t = \frac{\sqrt{\{(1 + x'^2 + y'^2)(1 + x^2 + y^2)\cos^2 r - (1 + xx' + yy')^2\}}}{(1 + xx' + yy')}.$$

* See the Appendix to Graves' translation of Chasles' *Memoirs on Cones and Spherical Conics*. x and y are the trigonometrical tangents of the intercepts made on the rectangular axes of reference by the perpendicular arcs drawn to them from the point.

Now the equation of the sphero-conic being $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$$y^2=\frac{b^2}{a^2}(a^2-x^2) \quad \text{and} \quad y=\frac{b}{a}\sqrt{a^2-x^2}.$$

Therefore the first power of y cannot enter into the expression for $\tan t$, if it is to be a rational linear function of x; that is, $y'=0$, or the centre of the circle lies on the major axis of the conic. So we have

$$(1+x'^2)(1+x^2+y^2)\cos^2 r-(1+xx')^2, \text{ a perfect square;}$$

or,

$$(1+x'^2)\left\{1+x^2+\frac{b^2}{a^2}(a^2-x^2)\right\}\cos^2 r-(1+xx')^2, \text{ a perfect square.}$$

Therefore,

$$(1+x'^2)(1+b^2)\frac{a^2-b^2}{a^2}\cos^2 r-x'^2(1+b^2)-\frac{a^2-b^2}{a^2}=0.$$

That is,

$$\frac{x'^2(1+a^2)}{(1+x'^2)(a^2-b^2)}+\frac{(1+b^2)}{b^2}\sin^2 r=1;$$

or,

$$\frac{\sin^2 r}{\sin^2 B}+\frac{\sin^2\mu}{\sin^2 C}=1,$$

where

$$\sin^2\mu=\frac{x'^2}{1+x'^2}, \quad \sin^2 B=\frac{b^2}{1+b^2}, \quad \sin^2 C=\frac{a^2-b^2}{1+a^2},$$

as the relation connecting the radius r of the circle with the abscissa μ of its centre.

Definition.—Any circle satisfying this condition is called a *focal circle* of the sphero-conic.

2. Supposing the relation,

$$\frac{\sin^2\mu}{\sin^2 C}+\frac{\sin^2 r}{\sin^2 B}=1$$

to be fulfilled, let us next inquire what will be the form of the expression under the radical sign—what, in fact, will be its square root.

From the equation

$$(1+x'^2)(1+b^2)\frac{a^2-b^2}{a^2}\cos r^2 - x'^2(1+b^2) - \frac{a^2-b^2}{a^2} = 0$$

we have

$$\cos^2 r = \frac{a^2x'^2(1+b^2)+a^2-b^2}{(1+x'^2)(1+b^2)(a^2-b^2)}.$$

Now, substituting this value of $\cos^2 r$ in the expression

$$\{(1+x'^2)(1+b^2)\cos^2 r - 1\} - 2xx' + x^2\left\{(1+x'^2)\frac{a^2-b^2}{a^2}\cos^2 r - x'^2\right\}$$

it reduces to

$$\frac{a^2x'^2(1+b^2)}{a^2-b^2} - 2xx' + \frac{a^2-b^2}{a^2(1+b^2)}x^2.$$

Denoting $\frac{a^2-b^2}{a^2(1+b^2)}$ by ϵ^2, we have, finally,

$$\tan t = \pm\frac{\frac{x'}{\epsilon} - \epsilon x}{1+xx'},$$

or,

$$t = \tan^{-1}\left(\frac{x'}{\epsilon}\right) - \tan^{-1}\epsilon x.$$

Cor. 1. The tangent arc from any point on the sphero-conic to the central focal ($x' = 0$) is $\tau = \tan^{-1}\epsilon x$.

Cor. 2. The tangent arc from the extremity of the axis minor ($x = 0$) to any focal (μ, r) is $t_0 = \tan^{-1}\frac{x'}{\epsilon}$.

Cor. 3. The tangent arc t from any point, x, y of the conic, to any focal (μ, r) may be expressed in the form $t = t_0 - \tau$.

Cor. 4. The locus of a point, such that the sum or difference of the tangent arcs from it to two circles is constant, is a sphero-conic of which the circles are focals.

Cor. 5. If a *point* be such that the trigonometrical tangent of the arc (t) drawn from it to any point of a sphero-conic is a rational function of the

trigonometrical tangent of the abscissa, then the abscissa is given by the equation,

$$x'^2 = \frac{a^2 - b^2}{1 + b^2} = a^2\epsilon^2,$$

and t by the equation,

$$\tan t = \frac{a - \epsilon x}{1 + a\epsilon x},$$

or,

$$t = A - \tan^{-1}(\epsilon x).$$

The points thus determined are the foci of the sphero-conic.

Cor. 6. Using the value of $\tan t$, we find

$$\operatorname{cosec}^2 t = \frac{\frac{x'^2}{\epsilon^2} + \epsilon^2 x^2 + 1 + x^2 x'^2}{\left(\frac{x'}{\epsilon} - \epsilon x\right)^2} = \frac{\left(\frac{x'^2}{\epsilon^2} + 1\right)(1 + \epsilon^2 x^2)}{\left(\frac{x'}{\epsilon} - \epsilon x\right)^2}.$$

But $1 + \epsilon^2 x^2 = \dfrac{(1 + x^2 + y^2)}{1 + b^2}$, since x, y is on the conic.

Therefore,

$$\sin t = \frac{\frac{x'}{\epsilon} - \epsilon x}{\sqrt{(1 + x^2 + y^2)\left(1 + \frac{x'^2}{\epsilon^2}\right)\left(\frac{1}{1 + b^2}\right)}}.$$

Hence, $\sin t$ bears a constant ratio to the sine of the perpendicular from x, y on the great circle

$$x = \frac{x'}{\epsilon^2},$$

which is a perpendicular to the major axis at an angular distance from the centre of the conic equal to

$$\tan^{-1}\left(\frac{x'}{\epsilon^2}\right).$$

Def.—This great circle is termed the *directrix* corresponding to the focal circle.

Cor. 7. The conic and focal have double contact (real or imaginary) at the points (real or imaginary) where the corresponding directrix meets them.

For, obviously, the tangent to the focal must vanish when the perpendicular on the directrix vanishes, therefore the directrix passes through all the points of intersection of the conic and circle, &c.

The demonstration of *Cors.* 6 and 7, for which I am indebted to Mr. T. Preston, is more elegant than that at which I originally arrived.

3. If, however, we start from the equation of the sphero-conic in the form

$$y^2 = \frac{2b^2}{a}x - \frac{b^2}{a^2}(1-a^2)x^2,$$

the origin being at the extremity of the axis major, then we must determine the relation between x', the tangent of the abscissa of the centre, and r, the radius of the focal circle, by making

$$(1+x'^2)\left[1+x^2+\frac{2b^2}{a}-\frac{b^2}{a^2}(1-a^2)x^2\right]\cos^2 r-(1+2xx'+x^2x'^2)$$

a perfect square.

Proceeding, as before, we find as the condition,

$$(1+x'^2)\left\{\frac{x'^2}{1+x'^2}-\sin^2 r\right\}=\left(x'-\frac{b^2}{a}\right)^2\frac{1}{(1+b^2)e^2};\ \text{where } e^2=\frac{a^2-b^2}{a^2}.$$

This equation is satisfied by

$$x'=a \text{ and } r=B\left(\text{or } \sin^2 r=\frac{b^2}{1+b^2}\right),$$

which give the central focal.

It is also satisfied by

$$r=0,\ x'=\tan(A\pm C),$$

which give the foci.

And, again, it brings into view the osculating circle at the end of the axis major, for it is satisfied by

$$x'=\frac{b^2}{a} \text{ and } \sin^2 r=\frac{x'^2}{1+x'^2}.$$

Extracting the square root of the above expression we have

$$\pm \tan t = \frac{\left[x'\frac{b^2}{a} - (e^2 + b^2)\right]x + \left(x' - \frac{b^2}{a}\right)}{e\sqrt{1 + b^2}\,(1 + xx')},$$

for the tangent of the arc drawn from any point of the conic to touch the focal.

If we make $x = 0$ we find t_0 the tangent from the extremity of the axis major to the focal, and as in the case of the central focal we may now express the tangent to any focal in terms of t_0, and the tangent τ to the osculating focal at the extremity of the axis major. Thus,

$$\pm t = t_0 \pm \tau,$$

the sign being determined by whether the point lies between the directrices of the circles or not. The equations which determine t_0 and τ are obviously derived from the above equation for t, by the substitution $x = 0$ and $x' = \frac{b^2}{a}$, respectively. Thus,

$$\pm t_0 = \tan^{-1}\left(\frac{x' - \frac{b^2}{a}}{e\sqrt{1 + b^2}}\right),$$

and

$$\pm \tau = \tan^{-1}\left(\frac{\left\{\frac{b^4}{a^2} - (e^2 + b^2)\right\}x}{e\sqrt{1 + b^2}\left(1 + \frac{b^2}{a}x\right)}\right) = \tan^{-1}\left\{\frac{-ex\sqrt{1 + b^2}}{1 + x\frac{b^2}{a}}\right\}.$$

Thus we may write the expression for $\tan t$ in the form,

$$\tan t = \frac{\dfrac{-ex\sqrt{1 + b^2}}{1 + x\frac{b^2}{a}} + \dfrac{x' - \frac{b^2}{a}}{e\sqrt{1 + b^2}}}{1 + x\dfrac{\left(x' - \frac{b^2}{a}\right)}{1 + x\frac{b^2}{a}}},$$

which shows, directly, that

$$\pm t = t_0 \pm \tau.$$

4. The following is a direct investigation of the converse problem :—

To find the locus of a point (P) on the surface of a sphere, such that the sine of the tangent arc (t), drawn from it to a given small circle, shall bear a constant ratio to the sine of the perpendicular arc (p) let fall from it upon a given great circle.

Take the spherical centre of the small circle for origin, and the perpendicular arc from it on the given great circle for axis of x. Then, if δ be the distance of P from the origin, and r the angular radius of the small circle, we have, if we denote the ordinate and abscissa, that is, the intercepts on these axes of reference by ξ and υ, respectively.

$$\cos t = \frac{\cos \delta}{\cos r}, \text{ and } \cos \delta = \cos \eta \cos \xi,$$

where η is the perpendicular arc from P on the axis of x. Therefore the equation

$$\sin t = m \sin p$$

becomes

$$\tan^2 \xi + \tan^2 \upsilon - \tan^2 r = m^2 (\sin^2 \beta - \sin 2\beta \tan \xi + \cos^2 \beta \tan^2 \xi),$$

if we use the equations

$$\sin p = \cos \eta \sin (\beta - \xi), \quad \text{and} \quad \tan \eta = \tan \upsilon \cos \xi,$$

where β is the angular distance of the given great circle from the centre of the small circle.

The locus is therefore a spherical conic, and the angular distance (μ) between its centre and the centre of the circle is given by the equation

$$\tan 2\mu = -\frac{m^2 \cos^2 r \sin 2\beta}{1 - m^2 \cos^2 r \cos 2\beta}.$$

Hence, if the centre of the conic be taken as origin, its equation becomes

$$\tan^2 \upsilon + \left\{1 - m^2 \cos^2 (\beta - \mu) - \frac{\sin^2 \mu}{\cos^2 r}\right\} \tan^2 \xi + \left\{1 - m^2 \sin^2 (\beta - \mu) - \frac{\cos^2 \mu}{\cos^2 r}\right\} = 0.$$

Comparing this with the standard form,

$$\frac{\tan^2 \xi}{\tan^2 A} + \frac{\tan^2 \upsilon}{\tan^2 B} = 1, \text{ or } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we have

$$\frac{\sin^2 r}{\sin^2 B} + \frac{\sin^2 \mu}{\sin^2 C} = 1.$$

The circle is therefore a focal circle of the sphero-conic.

5. The following proof, contributed by my friend Mr. T. Preston, will be thought interesting. Let the given great circle meet the small circle at A and B, and let tangents to the small circle at A and B meet at O. Then, if any tangent t be drawn from P to the circle, meeting OA at M, and the circle at T, and if the perpendicular arcs from P on OA, OB and AB be α, β, γ, respectively, while those from T on the same great circles are α', β', γ', we have, denoting $TM\,(= AM)$ by x,

$$\sin t \sin x \sin MAB + \sin x \sin \gamma = \sin (t + x) \sin \gamma';$$

or, since $\sin t$ bears a constant ratio (k) to $\sin \gamma$,

$$(1 + k \sin \phi) \frac{\sin \gamma}{\sin \gamma'} = \frac{\sin (t + x)}{\sin x} = \frac{\sin \alpha}{\sin \alpha'},$$

where k is the constant ratio of $\sin t$ to $\sin \gamma$, and ϕ the angle at which the great circle AB cuts the small circle.

Similarly, if PT meet OB at M' and if $TM'\,(= BM')$ be denoted by x', we have

$$(1 - k \sin \phi) \frac{\sin \gamma}{\sin \gamma'} = \frac{\sin (x' - t)}{\sin x'} = \frac{\sin \beta}{\sin \beta'}.$$

Hence,

$$(1 - k^2 \sin^2 \phi) \frac{\sin^2 \gamma}{\sin^2 \gamma'} = \frac{\sin \alpha \sin \beta}{\sin \alpha' \sin \beta'}.$$

But since T is on the small circle,

$$\sin^2 \gamma' = \sin \alpha' \sin \beta'.$$

Therefore,

$$\sin \alpha \sin \beta = \sin^2 \gamma (1 - k^2 \sin^2 \phi);$$

or the locus of P is a sphero-conic having *double contact* with the circle at the points where the great circle meets it.

6. To find the locus of a point (P) on the surface of a sphere, such that the sum of the tangent arcs (t and t') drawn from it to two given small circles shall be equal to a given constant ($2a$).

If δ and δ' be the angular distances of P from the spherical centres of the small circles, and if r and r' be the angular radii of the circles, we have

$$\cos t = \frac{\cos \delta}{\cos r} \quad \text{and} \quad \cos t' = \frac{\cos \delta'}{\cos r'}.$$

But if we take the arc joining the centres of the circles for axis of x, and the perpendicular arc at its middle point for axis of y, we have

$$\cos \delta = \cos \eta \cos (\gamma + \xi), \quad \text{and} \quad \cos \delta' = \cos \eta \cos (\gamma - \xi),$$

where 2γ is the angular distance between the centres of the circles, and η the perpendicular arc from P on it. Therefore,

$$\cos t = \frac{\cos \eta \cos (\gamma + \xi)}{\cos r}, \quad \text{and} \quad \cos t' = \frac{\cos \eta \cos (\gamma - \xi)}{\cos r'}.$$

Now, if t and t' be any two angles, we have the relation

$$\cos^2 t - 2 \cos t \cos t' \cos (t + t') + \cos^2 t' = \sin^2 (t + t').$$

Substituting for $\cos t$ and $\cos t'$, we find

$$\sin^2 2a + \frac{2 \cos 2a \cos^2 \eta \cos (\gamma+\xi) \cos (\gamma-\xi)}{\cos r \cos r'} - \cos^2 \eta \left\{ \frac{\cos^2 (\gamma+\xi)}{\cos^2 r} + \frac{\cos^2 (\gamma-\xi)}{\cos^2 r'} \right\} = 0.$$

Dividing by $\cos^2 \eta$, and noticing that $\tan \eta = \cos \xi \tan v$, this becomes

$$\begin{aligned} \sin^2 2a \,.\, \tan^2 v &+ \{ \sin^2 2a - (\sec^2 r + \sec^2 r' + 2 \cos 2a \sec r \sec r') \sin^2 \gamma \} \tan^2 \xi \\ &+ (\sec^2 r - \sec^2 r') \sin 2\gamma \tan \xi \\ &+ \sin^2 2a - (\sec^2 r + \sec^2 r' - 2 \cos 2a \sec r \sec r') \cos^2 \gamma = 0. \end{aligned}$$

This, being an equation of the second degree in $\tan\xi$ and $\tan v$, represents a spherical conic.

If the difference of the tangents $(t - t')$ had been given equal to 2α, we should have arrived at the same equation for the locus of P.

Let us now proceed to banish the term containing $\tan\xi$ by a change of origin. Instead of ξ let us write $\xi + \lambda$, and we find that the coefficient of $\tan\xi$ will vanish if

$$(L\cos^2\gamma - M\sin^2\gamma)\sin 2\lambda + N\sin 2\gamma\cos 2\lambda = 0,$$

where

$$L = \sec^2 r - 2\sec r\sec r'\cos 2\alpha + \sec^2 r',$$

$$M = \sec^2 r + 2\sec r\sec r'\cos 2\alpha + \sec^2 r',$$

$$N = \sec^2 r - \sec^2 r'.$$

The equation of the conic then becomes

$$P\tan^2 v + Q\tan^2\xi + R = 0, \qquad (E)$$

where

$$P = \sin^2 2\alpha,$$

$$Q = \sin^2 2\alpha - L\cos^2\gamma\sin^2\lambda - M\sin^2\gamma\cos^2\lambda - N\sin 2\gamma\sin\lambda\cos\lambda,$$

$$R = \sin^2 2\alpha - L\cos^2\gamma\cos^2\lambda - M\sin^2\gamma\sin^2\lambda + N\sin 2\gamma\sin\lambda\cos\lambda.$$

Now, comparing (E) with the standard form,

$$\frac{\tan^2\xi}{\tan^2 A} + \frac{\tan^2 v}{\tan^2 B} = 1,$$

we have

$$\frac{\sin^2(\gamma+\lambda)}{\cos^2 r} + \frac{2\sin(\gamma+\lambda)\sin(\gamma-\lambda)\cos 2\alpha}{\cos r\cos r'} + \frac{\sin^2(\gamma-\lambda)}{\cos^2 r'} = \frac{\sin^2 C}{\sin^2 A}\sin^2 2\alpha, \qquad (F)$$

$$\frac{\cos^2(\gamma+\lambda)}{\cos^2 r} - \frac{2\cos(\gamma+\lambda)\cos(\gamma-\lambda)\cos 2\alpha}{\cos r\cos r'} + \frac{\cos^2(\gamma-\lambda)}{\cos^2 r'} = \frac{\cos^2 C}{\cos^2 A}\sin^2 2\alpha, \qquad (G)$$

$$\frac{\sin 2(\lambda+\gamma)}{\cos^2 r} - \frac{2\sin 2\lambda\cos 2\alpha}{\cos r\cos r'} + \frac{\sin 2(\lambda-\gamma)}{\cos^2 r'} = 0. \qquad (H)$$

Now, equation (F) is equivalent to

$$\left\{\frac{\sin^2 C}{\sin^2 A} - \frac{\sin^2(\gamma+\lambda)}{\cos^2 r}\right\} \sin^2 2\alpha = \left\{\frac{\sin(\gamma+\lambda)\cos 2\alpha}{\cos r} + \frac{\sin(\gamma-\lambda)}{\cos r'}\right\}^2 = \Sigma^2,$$

and (G) is equivalent to

$$\left\{\frac{\cos^2 C}{\cos^2 A} - \frac{\cos^2(\gamma+\lambda)}{\cos^2 r}\right\} \sin^2 2\alpha = \left\{\frac{\cos(\gamma+\lambda)\cos 2\alpha}{\cos r} - \frac{\cos(\gamma-\lambda)}{\cos r'}\right\}^2 = \Gamma^2.$$

Multiplying, we have, with the use of equation (H),

$$\frac{\sin(\gamma+\lambda)\cos(\gamma+\lambda)}{\cos^2 r} \sin^2 2\alpha = \Sigma\Gamma,$$

or,

$$\frac{\sin^2 C}{\sin^2 A} \frac{\cos^2(\gamma+\lambda)}{\cos^2 r} + \frac{\cos^2 C}{\cos^2 A} \frac{\sin^2(\gamma+\lambda)}{\cos^2 r} = \frac{\sin^2 C \cos^2 C}{\sin^2 A \cos^2 A}.$$

Now, $\gamma+\lambda$ is the angular distance μ of the centre of one of the circles from the centre of the conic; hence the angular radius of either circle is connected with the angular distance of its centre from the centre of the spherical conic by the relation

$$\frac{\sin^2 r}{\sin^2 B} + \frac{\sin^2 \mu}{\sin^2 C} = 1;$$

that is, both the circles are focal circles of the conic.

Cor. $$\frac{\sin^2 2C}{\sin^2 2A} = \frac{(P-Q)(P-R)}{P^2} = \frac{(LM-N^2)\sin^2 2\gamma}{4P^2} = \frac{\sin^2 2\gamma}{\sin^2 2\alpha} \sec^2 r \sec^2 r'.$$

This is the theorem in spherical geometry which answers to the proposition, that when a plane conic is described as the locus of a point such that the sum or difference of the tangents drawn from it to two circles is given, the excentricity of the conic is independent of the radii of the circles, and is measured by the ratio of the distance between their centres to the given sum or difference of the tangents.

7. Properties of the Focal Circles of Sphero-Conics.

The following relations are analogous to those already proved for the

focal circles of plane conics, and may be demonstrated in a similar manner:—

(α) Any great circle chord (AB) of a sphero-conic, touching a focal circle at C, and meeting the corresponding directrix in D, is cut harmonically in A, B, C, D.

(β) If AB and $A'B'$ be any two chords touching a focal circle at C and C', the great circles AA', BB', CC' meet on the corresponding directrix.

(γ) As a particular case of (β) it follows that, if from any point A of a conic two tangent arcs, AB and AB', be drawn to a focal circle, the tangent arc at A to the conic passes through the point of intersection of BB' and CC', which is a point on the directrix.

We have, therefore, a simple method of drawing a tangent at any point of a conic when we are given one of its focal circles.

(δ) The polar of any point on the directrix, with reference to the conic, is identical with its polar with reference to the focal circle; and the pole of the directrix, with reference to the conic, coincides with its pole with reference to the circle; or the curves have double contact at the points (real or imaginary) where the directrix meets them.

(ϵ) The intersection of the tangents at A and A', the intersection of the tangents at B and B', the intersection of the great circles AB and $A'B'$, and of the great circles AB' and $A'B$, and the points of contact of the tangents from R to the circle, are all on the same great circle, viz. the polar of R with reference to the conic or focal.

Thus, of the three pairs of great circles joining the four points, A, B, A', B', one pair meets at R on the directrix, and the other two pairs intersect on the polar of R.

8. If t and t' be the tangent arcs from any point (P) of a sphero-conic to a pair of its focal circles, δ and δ' the angular distances of the point from the spherical centres of the circles, we have, if r and r' be their radii,

$$\cos t = \frac{\cos \delta}{\cos r'}, \quad \therefore dt = \frac{\sin \delta d\delta}{\sin t \cos r}.$$

But

$$\frac{\sin t \cos r}{\sin \delta} = \cos \tfrac{1}{2}\alpha, \quad \therefore dt = \frac{d\delta}{\cos \tfrac{1}{2}\alpha};$$

where α is the angle the circle subtends at the point (P) from which the tangents are drawn.

Similarly,
$$dt' = \frac{d\delta'}{\cos\frac{1}{2}\alpha'}.$$

Now, if the tangent arc to the conic at P make angles θ and θ' with the arcs δ and δ', we have, denoting the element of the conic by ds,

$$\cos\theta = \frac{d\delta}{ds} \quad \text{and} \quad \cos\theta' = \frac{d\delta'}{ds}.$$

But, since $t \pm t' = \text{const.}$, we have $dt \pm dt' = 0$; therefore

$$\frac{\cos\theta}{\cos\frac{1}{2}\alpha} \pm \frac{\cos\theta'}{\cos\frac{1}{2}\alpha'} = 0,$$

a theorem for spherical conics which also holds *in plano.*

Again, if the perpendicular arcs from the centres of the circles on the tangent at P make intercepts T and T' on it, measured from the point of contact, we have

$$\tan T = \tan\delta\cos\theta, \quad \text{and} \quad \tan T' = \tan\delta'\cos\theta',$$

also,

$$\tan t = \tan\delta\cos\tfrac{1}{2}\alpha, \quad \text{and} \quad \tan t' = \tan\delta'\cos\tfrac{1}{2}\alpha'.$$

Therefore, by the above relation we have

$$\frac{\tan T}{\tan t} \pm \frac{\tan T'}{\tan t'} = 0.$$

Cor. The tangent at any point of a sphero-conic makes equal angles with the focal radii to that point.

9. In what precedes we have proved the following propositions:—

(1.) If the sine of the arc, drawn from a point P on the surface of the sphere, to touch a fixed small circle (μ, r), bears a constant ratio to the sine of the arc let fall from it perpendicularly on a fixed great circle, the locus of the point P will be a sphero-conic.

The major diameter of the sphero-conic will pass through the centre of

the fixed circle, and will be perpendicular to the fixed great circle, and the fixed small circle will be related to the conic by the equation,

$$\frac{\sin^2 r}{\sin^2 B} + \frac{\sin^2 \mu}{\sin^2 C} = 1.$$

That is to say, it will be a focal circle of the sphero-conic. In this equation,

μ = angular distance between centre of circle and centre of conic.

r = angular radius of the small circle.

$2B$ = minor diameter of the sphero-conic.

$2C$ = the angular distance between its foci.

The Corresponding Supplemental Theorem is:—

A great circle will envelope a sphero-conic if the sine of the angle which it makes with a fixed circle bears a constant ratio to the sine of the arc let fall perpendicularly on it from a fixed point.

The minor diameter of the sphero-conic will pass through the centre of the circle, and the circle itself will be related to the conic by the equation

$$\frac{\cos^2 r}{\cos^2 A} + \frac{\cos^2 \mu}{\cos^2 \phi} = 1,$$

in which

μ = angular distance between the centres of the circle and conic.

r = spherical radius of the circle.

$2A$ = the major diameter of the sphero-conic.

2ϕ = the angle between its cyclic arcs.

(II.) The locus of a point on the sphere will be a sphero-conic if the sum (or difference) of the tangent arcs drawn from it to two fixed circles is constant.

The major diameter of the conic passes through the centres of the two fixed circles, and they will be related to it by the same equation,

$$\frac{\sin^2 r}{\sin^2 B} + \frac{\sin^2 \mu}{\sin^2 C} = 1,$$

in which μ, r, B, C have the same signification as before.

The Corresponding Supplemental Theorem is :—

If an arc makes, with two fixed circles, angles, the sum or difference of which is constant, it will envelope a sphero-conic;

The minor diameter of the conic passes through the centres of the two fixed circles; and they will be related to it by the same equation,

$$\frac{\cos^2 r}{\cos^2 A} + \frac{\cos^2 \mu}{\cos^2 \phi} = 1,$$

in which μ, ϕ, r, A have the same signification as before.

Note to Cor. 3, *Art.* 2.—The following interesting note, supplied by my friend Mr. T. Preston, enables us to take a perfectly clear view of the manner in which the tangent to a focal changes sign as the point from which it is drawn crosses the corresponding directrix.

The equation,

$$t = t_0 - \tau$$

is very suggestive, and it is worthy of remark that the tangent t, from a variable point on the conic to a focal, changes sign as the point passes through the point of contact of the conic and focal. Thus, if the centre of the focal be situated in the right-hand half of the conic, and if the point x, y, from which t is drawn, be in the left-hand half, then the equation which determines t is

$$t = t_0 + \tau, \qquad (1)$$

for in this case x is negative and x' is positive, and the equation for t becomes

$$\tan t = \frac{x'/\epsilon + \epsilon x}{1 - xx'},$$

when x, y passes the extremity of the axis-minor, τ changes sign, and the equation becomes

$$t = t_0 - \tau; \qquad (2)$$

and as x, y moves towards the directrix, t diminishes, and ultimately vanishes when x, y coincides with the point of contact of the conic and focal.

When x, y passes through the point of contact, t changes sign, and the equation becomes

$$-t = t_0 - \tau. \qquad (3)$$

Now, if we suppose the circle to vary, and if we consider the tangents t_1 and t_2 drawn to it from any two fixed points of the conic, the equations (1) and (2), or (3), show that the difference of t_1 and t_2 will be constant as the circle moves, if the points are on the same side of the directrix, and the sum of t_1 and t_2 will be constant if the points lie on opposite sides of the directrix.

Hence the sum or difference of the tangents drawn from any two fixed points of a conic plane or spherical to any variable focal circle will be constant, according as the points lie on opposite sides, or the same side, of the corresponding directrix.

When the variable point, x, y coincides with one of the contact points of the focal $t = 0$, and consequently $t_0 = \tau$, or the tangent to the central focal from one of the contact points of any focal is equal to the tangent to that focal from one of the contact points of the central focal.

This theorem holds generally for any two focal circles of a plane or spherical conic. For the tangent t, from one of the contact points of a focal A to any other focal B, is given by

$$t = t_0 - \tau,$$

if A and B both lie on the right-hand, or positive side, of the conic, and if A lies between B and the centre. Similarly, the tangent to A from a contact point of B is given by

$$-t' = t_0' - \tau';$$

therefore,

$$t - t' = t_0 - \tau' + t_0' - \tau = 0,$$

since $t_0 = \tau'$ and $t_0' = \tau$ by the particular theorem just stated. Hence $t = t'$ or *the tangents to any focal A from the contact points of any other focal B, are equal to the tangents to B from the contact points of A.*

APPENDIX.

My friend Mr. Preston has followed up the track upon which I entered when I discussed the focal circle whose centres lie on the axis major, by investigating the corresponding theorems respecting the focals whose centres lie on the transverse axis. I had not failed to consider these focals. But I abstained from discussing them in detail, because many of their properties involve imaginaries which did not appear to me to admit at all readily of geometrical interpretation. His results, however, are so interesting in their form, that I have obtained leave from him to introduce them into the following Appendix :—

The problem of the first Article might equally have been : "How far is a circle limited, if the trigonometrical tangent of the arc, drawn to touch it from any point on a sphero-conic, shall always be a rational linear function of the trigonometrical tangent of the ordinate (y) of the point?"

In this case we have

$$(1 + x'^2 + y'^2)(1 + x^2 + y^2)\cos^2 r - (1 + xx' + yy')^2,$$

a perfect square, subject to the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence, $x' = 0$, or the centre of the circle must lie on the axis of y; and if, as before, we denote the angular distance between the centre of the conic and the centre of the circle by μ, we have the further relation between μ and the radius r of the circle,

$$\frac{\sin^2 r}{\sin^2 A} - \frac{\sin^2 \mu}{\tan^2 C} = 1,$$

where $2C$, as before, is the angular distance between the foci of the conic. Using this relation, the expression for $\tan t$ reduces to

$$\tan^2 t = -\left\{ y^2 \frac{a^2 - b^2}{b^2(1 + a^2)} + 2yy' + y'^2 \frac{b^2(1 + a^2)}{a^2 - b^2} \right\} \frac{1}{(1 + yy')^2};$$

or, if we denote

$$\sqrt{\frac{a^2 - b^2}{b^2(1+a^2)}} \text{ by } e',$$

we have

$$\tan t = \frac{i\left(ye' + \frac{y'}{e'}\right)}{1 + yy'}, \text{ where } i = \sqrt{-1}.$$

That is,

$$t = \tan^{-1}(ie'y) + \tan^{-1}\left(\frac{iy'}{e'}\right).$$

Hence the focal circle has external double contact with the conic; for since t is imaginary the conic lies wholly within the circle.

Cor. Using the foregoing value of tan t, we have

$$\sin^2 t = \frac{\left(ye' + \frac{y'}{e'}\right)^2}{\left(\frac{y'^2}{e'^2}\right)(1 - y^2 e'^2)}.$$

But

$$1 + x^2 + y^2 = (1 + a^2)(1 - y^2 e'^2),$$

since x, y is on the conic. Therefore sin t is proportional to the sine of the perpendicular (p) from the point x, y on the great circle

$$ye' + \frac{y'}{e'} = 0,$$

which is the *directrix* corresponding to the focal circle.

Hence, generally, the locus of a point, such that the sine of the tangent arc drawn from it to a fixed small circle bears a constant ratio to the sine of the perpendicular arc from it to a fixed great circle, is a sphero-conic, of which the small circle is a focal of the major axis system, or minor axis system, according as the given ratio is real or imaginary.

The two systems of focal circles may be discussed as the corresponding systems *in plano.*